Some New Exact Ground States for Generalized Hubbard Models

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Abstract

A set of new exact ground states of the generalized Hubbard models in arbitrary dimensions with explicitly given parameter regions is presented. This is based on a simple method for constructing exact ground states for homogeneous quantum systems.

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There has been a growing interest in constructing exact ground states for correlated Fermion systems in certain parameter regions [1-7]. For one dimensional quantum systems, an effective method to construct ground states is the so called matrix product approach [1]. Its origin can be traced back to the $S = 1$ spin chain model [2]. For higher dimensional correlated Fermion systems, there are also some approaches for the construction of exact ground states [3]. In [4] the conditions for η-pairing, paramagnetic, Néel, charge-density-wave and ferromagnetic states to be exact ground states of the generalized Hubbard model have been presented by using the so called optimal ground state approach. These approaches recovered and improved the results obtained previously.

In this paper we give some new exact ground states for the generalized Hubbard models. We first describe a general way to construct some exact ground states for homogeneous quantum systems. Let H be the Hamiltonian of a quantum lattice system (correlated Fermion system) of the form $H = \sum_{<ij>} h_{ij}$, where $< ij >$ denotes the nearest neighbours, i, j are points on a finite lattice Γ with Λ lattice sites, H acts in the tensor space $\mathcal{H}$ associated with Γ (i.e. $\mathcal{H}$ is the Λ-fold power of some given Hilbert space K associated with a single lattice point), h_{ij} is a self-adjoint operator acting on the tensor product $K \otimes K$ associated with i, j. We call the system homogeneous if the spectral decomposition of h_{ij} is independent of i, j. For square lattice case, this homogeneity implies that the horizontal and vertical couplings are the same. The exact ground states of H are by definition the eigenstates belonging to the lowest eigenvalue of H. Let $|\xi_\alpha >$, $\alpha = 1, ..., n$ be n possible states of a local site, for instance, empty, spin up (down) and double occupied states. We have, for some given indices $\alpha, \beta \in 1, ..., n$, the following conclusions:

I) If $|\xi_\alpha >_i |\xi_\alpha >_j$ is an eigenvector of h_{ij} and the corresponding eigenvalue is the lowest one, the following state is a ground state of the Hamiltonian H of the system,

$$\psi^I = \prod_{i=1}^{\Lambda} |\xi_\alpha >_i . \tag{1}$$

II) If $|\xi_\alpha >_i |\xi_\alpha >_j$, $|\xi_\beta >_i |\xi_\beta >_j$ and $|\xi_\alpha >_i |\xi_\beta >_j +|\xi_\beta >_i |\xi_\alpha >_j$ are eigenvectors of h_{ij} with the same and lowest energy, the state

$$\psi^{II} = \prod_{i=1}^{\Lambda} (|\xi_\alpha >_i +|\xi_\beta >_i) \tag{2}$$

is a ground state of H.

III) If $|\xi_\alpha >_i |\xi_\alpha >_j$ and $|\xi_\alpha >_i |\xi_\beta >_j -|\xi_\beta >_i |\xi_\alpha >_j$ are eigenvectors of h_{ij} with the same and lowest energy, the state

$$\psi^{III} = \sum_{i=1}^{\Lambda}(-1)^{i+1}|\xi_\alpha >_1 |\xi_\alpha >_2 ...|\xi_\beta >_i ...|\xi_\alpha >_\Lambda \tag{3}$$

is a ground state of H.

IV) If $|\xi_\alpha >_i |\xi_\alpha >_j$ and $|\xi_\alpha >_i |\xi_\beta >_j +|\xi_\beta >_i |\xi_\alpha >_j$ are eigenvectors of h_{ij} with the same and lowest energy, the state

$$\psi^{IV} = \sum_{i=1}^{\Lambda} |\xi_\alpha >_1 |\xi_\alpha >_2 ...|\xi_\beta >_i ...|\xi_\alpha >_\Lambda \tag{4}$$

is a ground state of H.

[Proof]. I is a consequence of homogeneity, II holds by a simple calculation. As for III (IV can be similarly proved) we proceed as follows. Let

$$A_i = \begin{pmatrix} |\xi_\alpha >_i & 0 \\ |\xi_\alpha >_i +|\xi_\beta >_i & -|\xi_\alpha >_i \end{pmatrix}.$$

We have

$$A_i A_j = \begin{pmatrix} |\xi_\alpha >_i |\xi_\alpha >_j & 0 \\ |\xi_\alpha >_i |\xi_\beta >_j -|\xi_\beta >_i |\xi_\alpha >_j & |\xi_\alpha >_i |\xi_\alpha >_j \end{pmatrix}.$$

The nonzero elements a_{ij} of $A_i A_j$ satisfy $h_{ij}a_{ij} = \epsilon a_{ij}$, with ϵ the lowest eigenvalue of h_{ij}. Therefore the elements of the matrix $M \equiv \prod_{i=1}^{\Lambda} A_i$, as well as their linear combinations are ground states of H with the same lowest eigenvalue $P\epsilon$, where P is the number of nearest-neighbour pairs of the lattice. The linear combination of the elements $M_{(1,1)}$ and $M_{(2,1)}$ gives rise to the ground state ψ^{III}. ∎

The state ψ^{II} is a sum of terms containing factors of the form $|\xi_\alpha >_i$ and $|\xi_\beta >_j$. If there exists a local operator o such that $o|\xi_\alpha >= a_1|\xi_\alpha >$, $o|\xi_\beta >= a_2|\xi_\beta >$, $a_1 \neq a_2$ and the operator $O = \sum_i^{\Lambda} o_i$ commutes with the Hamiltonian H, then

$$\psi_l^{II} = \sum_{m_1,...,m_l=1}^{\Lambda} |\xi_\alpha >_1 ...|\xi_\beta >_{m_1} ...|\xi_\beta >_{m_l} ...|\xi_\alpha >_\Lambda \tag{5}$$

are also ground states for all $l = 0, 1, ..., \Lambda$. In this case the states $\psi_{l=1,\Lambda-1}^{II}$ (resp. $\psi_{l=0,\Lambda}^{II}$) are just the states ψ^{IV} (resp. ψ^I). However, the state ψ^{IV} (resp. ψ^I) has a larger parameter region than ψ^{II}. In addition, all the states ψ_l^{II} are degenerate.

We now consider the generalized Hubbard model on an arbitrary d-dimensional lattice with Hamiltonian [6],

$$
\begin{aligned}
H &= \sum_{<ij>}^{\Lambda} h_{ij} \equiv \sum_{<ij>}^{\Lambda} \Big[-t \sum_{\sigma} (c_{i\sigma}^{\dagger} c_{j\sigma} + c_{j\sigma}^{\dagger} c_{i\sigma}) + X \sum_{\sigma} (c_{i\sigma}^{\dagger} c_{j\sigma} + c_{j\sigma}^{\dagger} c_{i\sigma})(n_{i,-\sigma} + n_{j,-\sigma}) \\
&\quad + \tfrac{U}{Z} \Big(\big(n_{i\uparrow} - \tfrac{1}{2} \big)\big(n_{i\downarrow} - \tfrac{1}{2} \big) + \big(n_{j\uparrow} - \tfrac{1}{2} \big)\big(n_{j\downarrow} - \tfrac{1}{2} \big) \Big) + V(n_i - 1)(n_j - 1) \\
&\quad + Y(c_{i\uparrow}^{\dagger} c_{i\downarrow}^{\dagger} c_{j\downarrow} c_{j\uparrow} + c_{j\uparrow}^{\dagger} c_{j\downarrow}^{\dagger} c_{i\downarrow} c_{i\uparrow}) + \tfrac{J_{xy}}{2}(S_i^+ S_j^- + S_j^+ S_i^-) + J_z S_i^z S_j^z + \tfrac{\mu}{Z}(n_i + n_j) \Big],
\end{aligned}
\tag{6}
$$

where $c_{i\sigma}^{\dagger}$ and $c_{i\sigma}$, $\sigma = \uparrow, \downarrow$, are canonical Fermi creation resp. annihilation operators, $n_{i,\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$ and $n_i = n_{i\uparrow} + n_{i\downarrow}$ are the number operators, $S_i^+ = c_{i\uparrow}^{\dagger} c_{i\downarrow}$, $S_i^- = c_{i\downarrow}^{\dagger} c_{i\uparrow}$, $S_i^z = (n_{i\uparrow} - n_{i\downarrow})/2$; μ is the chemical potential and Z the coordination number of the d-dimensional lattice; t, X, U, V, Y, J_{xy} and J_z are real valued coupling constants for, respectively, single particle hopping, bond-charge interaction, on-site Coulomb interaction, nearest-neighbour Coulomb interaction, pair-hopping and XXZ-type spin interactions.

The well known states related to the Hubbard model are the η-pairing, paramagnetic, Néel, charge-density-wave (CDW) and ferromagnetic (F) states, defined respectively as follows:

$$
\begin{aligned}
&|N\acute{e}el> = \prod_{i \in B} c_{i\uparrow}^{\dagger} \prod_{i \in B'} c_{i\downarrow}^{\dagger} |0>, \quad |\psi_m^{\eta}> = \Big(\sum_{j=1}^{\Lambda} e^{i\eta j} c_{j\downarrow}^{\dagger} c_{j\uparrow}^{\dagger} \Big)^m |0>, \quad m \in \mathbb{N} \\
&|para> = \prod_{i \in A} c_{i\uparrow}^{\dagger} \prod_{i \in A'} c_{i\downarrow}^{\dagger} |0>, \quad |CDW> = \prod_{i \in B} c_{i\uparrow}^{\dagger} c_{i\downarrow}^{\dagger} |0>, \quad |F> = \prod_{i} c_{i\uparrow}^{\dagger} |0>,
\end{aligned}
\tag{7}
$$

where $|0>$ is the vacuum state, $A \cap A' = \emptyset$, A and A' together span the whole lattice, B and B' are the odd and even sublattices of a bipartite lattice, η is a real parameter. The η-pairing states display off diagonal long-range order (ODLRO) [8], i.e. superconductivity (Meissner effect and flux quantization [9]).

The exact ground states of H depend on the coupling constants. At certain parameter regions the above states (7) become exact ground states [3, 4], as also follows from our conclusions. We observe that some of the ground states could be physically equivalent owing to some symmetries of the system in certain parameter regions, for instance, when $J_z = J_{xy}$, H is $SU(2)$ invariant. Nevertheless besides the well known exact ground states the quantum system (6) has many additional exact ground states of types ψ^{II}, ψ^{III} and ψ^{IV} with respect to different parameter regions. In the following we simply list some of these new ground states.

We first present some exact ground eigenstates of the form ψ^{III}.

$$|\psi_1^\sigma> = N \sum_{i=1}^{\Lambda} (-1)^{i+1} c_{1\sigma}^\dagger c_{2\sigma}^\dagger ... c_{i,-\sigma}^\dagger ... c_{\Lambda\sigma}^\dagger |0>, \quad N \equiv (\frac{1}{2})^{\frac{\Lambda}{2}} \tag{8}$$

is a ground state when $t = X$, $J_z = -J_{xy} < 0$, $\frac{U}{Z} \geq max(\frac{J_z}{4} + \frac{2\mu}{Z} - V, \frac{J_z}{2} + \frac{2\mu}{Z} + 2|t|, \frac{J_z}{4} + V + Y, \frac{J_z}{2} - \frac{2\mu}{Z} + 2|t|, \frac{J_z}{4} - \frac{2\mu}{Z} - V, \frac{J_z}{4} + V - Y)$. The eigenvalue is $(-\frac{U}{2Z} + \frac{J_z}{4} + \frac{2\mu}{Z})P$. If one sets $J_z = J_{xy} = 0$ instead of $J_z = -J_{xy} < 0$, state (8) is still a ground state but is then equivalent to the $|para>$ state, because $|\psi_1^\sigma>$ is a linear combination of $|para>$ states, while under condition $t = X$, $J_z = J_{xy} = 0$, any $|para>$ state is an exact ground state.

The state

$$|\psi_2^\sigma> = \sum_{i=1}^{\Lambda} (-1)^{i+1} c_{1\sigma}^\dagger c_{2\sigma}^\dagger ... c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger ... c_{\Lambda\sigma}^\dagger |0> \tag{9}$$

is a ground state at parameter region $X + \frac{\mu}{Z} \leq 0$ if $t \geq 0$, $X + \frac{\mu}{Z} \leq t$ if $t \leq 0$, $J_z = \frac{2U}{Z} + \frac{4\mu}{Z} - 4(t - 2X)$ and $max(-t + 2X + \frac{3\mu}{Z} - V, -t + 2X + V + Y + \frac{\mu}{Z}, -t + 2X - V - \frac{\mu}{Z}) \leq \frac{U}{2Z} < V - Y + \frac{\mu}{Z} - t + 2X$, $\frac{J_{xy}}{4} \geq max(\frac{U}{2Z} - t + 2X + \frac{\mu}{Z}, t - 2X - \frac{\mu}{Z} - \frac{U}{2Z} + 2(X - t)^2/(V - Y - t + 2X + \frac{\mu}{Z} - \frac{U}{2Z}))$, or $\frac{U}{2Z} \geq max(-t + 2X + \frac{3\mu}{Z} - V, -t + 2X + V + Y + \frac{\mu}{Z}, -t + 2X - V - \frac{\mu}{Z})$, $\frac{U}{2Z} > V - Y + \frac{\mu}{Z} - t + 2X$, $t - 2X - \frac{\mu}{Z} - \frac{U}{2Z} + 2(X - t)^2/(V - Y - t + 2X + \frac{\mu}{Z} - \frac{U}{2Z}) \geq \frac{J_{xy}}{4} \geq \frac{U}{2Z} - t + 2X + \frac{\mu}{Z}$ for $t \geq 2X$, $t \neq X$, or $max(t - V + \frac{3\mu}{Z}, t + \frac{\mu}{Z} + V + Y, t - V - \frac{\mu}{Z}, t + V + \frac{\mu}{Z} - Y) \leq \frac{U}{2Z} \leq min(\frac{J_{xy}}{4} - t - \frac{\mu}{Z}, -\frac{J_{xy}}{4} - t - \frac{\mu}{Z})$, $J_z = \frac{2U}{Z} + \frac{4\mu}{Z} + 4t$, $\mu \leq 0$ for $t = X \leq 0$. The eigenvalue of $|\psi_2^\sigma>$ is $(2X - t + \frac{3\mu}{Z})P$.

For $t \leq 0$ and $J_z = 4(t + \frac{U}{2Z} - \frac{\mu}{Z})$ the following state

$$|\psi_3^\sigma> = N \sum_{i=1}^{\Lambda} (-1)^{i+1} c_{1,\sigma}^\dagger c_{2,\sigma}^\dagger ... \mathbf{1}_i ... c_{\Lambda,\sigma}^\dagger |0>, \tag{10}$$

$\mathbf{1}$ being the identity operator, is an exact ground state, for $t \neq X$, $\frac{\mu}{Z} \geq X \geq t - \frac{\mu}{Z}$ and $V - Y + t - \frac{\mu}{Z} > \frac{U}{2Z} \geq max(t + \frac{\mu}{Z} - V, t - \frac{\mu}{Z} + V + Y, t - V - \frac{3\mu}{Z})$, $\frac{J_{xy}}{4} \geq max(\frac{U}{2Z} + t - \frac{\mu}{Z}, \frac{\mu}{Z} - t - \frac{U}{2Z} + 2(X - t)^2/(V - Y + t - \frac{U}{2Z} - \frac{\mu}{Z}))$ or $\frac{U}{2Z} \geq max(t + \frac{\mu}{Z} - V, t - \frac{\mu}{Z} + V + Y, t - V - \frac{3\mu}{Z})$, $\frac{U}{2Z} > V - Y + t - \frac{\mu}{Z}$, $\frac{\mu}{Z} - t - \frac{U}{2Z} + 2(X - t)^2/(V - Y + t - \frac{U}{2Z} - \frac{\mu}{Z}) \geq \frac{J_{xy}}{4} \geq \frac{U}{2Z} + t - \frac{\mu}{Z}$, or for $t = X \leq 0$, $\mu \geq 0$ and $max(t + \frac{\mu}{Z} - V, t - \frac{\mu}{Z} + V + Y, t - \frac{\mu}{Z} + V - Y) \leq \frac{U}{2Z} \leq min(-t + \frac{\mu}{Z} + \frac{J_{xy}}{4}, -t - \frac{J_{xy}}{4} + \frac{\mu}{Z})$.

Another kind of exact ground states of the generalized Hubbard model of the form ψ^{IV} is given by:

$$|\psi_4^\sigma> = \sum_{i=1}^{\Lambda} c_{1\sigma}^\dagger c_{2\sigma}^\dagger ... c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger ... c_{\Lambda\sigma}^\dagger |0> . \tag{11}$$

The corresponding parameter region is, for $t \leq 2X$, $t \neq X$, $J_z = 4(\frac{U}{2Z} + \frac{\mu}{Z} + t - 2X)$, $X - \frac{\mu}{Z} \geq t$ if $t \geq 0$, $X - \frac{\mu}{Z} \geq 0$ if $t \leq 0$, $\frac{U}{2Z} \geq max(t - 2X - V + \frac{3\mu}{Z}, \ t - 2X + V + Y + \frac{\mu}{Z}, \ t - 2X - \frac{\mu}{Z} - V)$ and $\frac{U}{2Z} < V - Y + t - 2X + \frac{\mu}{Z}$, $\frac{J_{xy}}{2} \geq max(\frac{U}{Z} + 2t - 4X + \frac{2\mu}{Z}, \ 4X - 2t - \frac{U}{Z} - \frac{2\mu}{Z} + 4(X - t)^2/(V - Y + t - 2X - \frac{U}{2Z} + \frac{\mu}{Z}))$ or $\frac{U}{2Z} > V - Y + t - 2X + \frac{\mu}{Z}$, $4X - 2t - \frac{U}{Z} - \frac{2\mu}{Z} + 4(X - t)^2/(V - Y + t - 2X - \frac{U}{2Z} + \frac{\mu}{Z}) \geq \frac{J_{xy}}{2} \geq \frac{U}{Z} + 2t - 4X + \frac{2\mu}{Z}$, and for $t = X \geq 0$, $J_z = 4(\frac{U}{2Z} + \frac{\mu}{Z} - t)$, $\mu \leq 0$, $min(\frac{J_{xy}}{4} + t - \frac{\mu}{Z}, t - \frac{J_{xy}}{4} - \frac{\mu}{Z}) \geq \frac{U}{2Z} \geq max(-t + V + Y + \frac{\mu}{Z}, -t - \frac{\mu}{Z} - V, \ -t + V - Y + \frac{\mu}{Z})$. $|\psi_6^\sigma>$ corresponds to the eigenvalue $(t - 2X + \frac{3\mu}{Z})P$.

For $t \geq 0$ and $J_z = 4(-t + \frac{U}{2Z} - \frac{\mu}{Z})$ the state

$$|\psi_5^\sigma> = N \sum_{i=1}^{\Lambda} c_{1,\sigma}^\dagger c_{2,\sigma}^\dagger ... \mathbf{1}_i ... c_{\Lambda,\sigma}^\dagger |0> \tag{12}$$

is an exact ground state, for $t \neq X$, $-\frac{\mu}{Z} \leq X \leq \frac{\mu}{Z} + t$, and $V - Y - t - \frac{\mu}{Z} > \frac{U}{2Z} \geq max(-t + \frac{\mu}{Z} - V, \ -t - \frac{\mu}{Z} + V + Y, \ -t - V - \frac{3\mu}{Z})$, $\frac{J_{xy}}{4} \geq max(\frac{U}{2Z} - t - \frac{\mu}{Z}, \ \frac{\mu}{Z} + t - \frac{U}{2Z} + 2(X - t)^2/(V - Y - t - \frac{U}{2Z} - \frac{\mu}{Z}))$ or $\frac{U}{2Z} \geq max(-t + \frac{\mu}{Z} - V, \ -t - \frac{\mu}{Z} + V + Y, \ -t - V - \frac{3\mu}{Z})$, $\frac{U}{2Z} > V - Y - t - \frac{\mu}{Z}$, $\frac{\mu}{Z} + t - \frac{U}{2Z} + 2(X - t)^2/(V - Y - t - \frac{U}{2Z} - \frac{\mu}{Z}) \geq \frac{J_{xy}}{4} \geq \frac{U}{2Z} - t - \frac{\mu}{Z}$, or for $t = X$, $\mu \geq 0$, and $max(-t + \frac{\mu}{Z} - V, \ -t - \frac{\mu}{Z} + V + Y, \ -t - \frac{\mu}{Z} + V - Y) \leq \frac{U}{2Z} \leq min(t + \frac{\mu}{Z} + \frac{J_{xy}}{4}, \ t - \frac{J_{xy}}{4} + \frac{\mu}{Z}$. The eigenvalue is $(-t + \frac{\mu}{Z})P$.

The states of the form ψ^{II} are sums of states with different number of electrons. As the Hamiltonian commutes with the number operator, any projection of a ground state of the type ψ^{II} to a state with a given number of electrons is also a ground state. These ground states are of the form (5), for instance, for $l = 2, ..., \Lambda - 2$, the following states are ground states,

$$|\psi_l^\sigma> = \sum_{m_1,...,m_l=1}^{\Lambda} c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger ... c_{m_1\sigma}^\dagger ... c_{m_l\sigma}^\dagger ... c_{\Lambda\uparrow}^\dagger c_{\Lambda\downarrow}^\dagger |0> \tag{13}$$

if the parameters satisfy $J_z = 4(\frac{U}{2Z} - t + \frac{\mu}{Z})$, $V = -\frac{\mu}{Z} - t - \frac{U}{2Z}$, $\mu \leq 0$, $|Y| - 2t \leq \frac{U}{Z} \leq 2t - \frac{|J_{xy}|}{2} - \frac{2\mu}{Z}$ for $t = X > 0$. For $t \neq X$ we have $J_z = 4(\frac{U}{2Z} + t - 2X + \frac{\mu}{Z})$, $V = -\frac{\mu}{Z} + t - 2X - \frac{U}{2Z}$, $\mu \leq 0$, $X \geq max(\frac{\mu}{Z}, t + \frac{\mu}{Z}, \frac{t}{2})$ and $2t - 4X + Y \leq \frac{U}{Z} < 2t - 4X - Y$, $\frac{J_{xy}}{2} \geq max(\frac{U}{Z} + 2t - 4X + \frac{2\mu}{Z}, \ 4X - 2t - \frac{2\mu}{Z} - \frac{U}{Z} + 4(X - t)^2/(2t - 4X - Y - \frac{U}{Z}))$ or $2t - 4X + Y \leq \frac{U}{Z}$, $2t - 4X - Y < \frac{U}{Z}$, $4X - 2t - \frac{2\mu}{Z} - \frac{U}{Z} + 4(X - t)^2/(2t - 4X - Y - \frac{U}{Z}) \geq \frac{J_{xy}}{2} \geq \frac{U}{Z} + 2t - 4X + \frac{2\mu}{Z}$. The eigenvalues are $(\frac{3\mu}{Z} + t - 2X)P$. For $l = 0, 1, \Lambda - 1, \Lambda$, the states $|\psi_l^\sigma>$ are already included in ψ^I and ψ^{IV}.

Similarly we have the following ground states:

$$|\phi_l^\sigma> = \sum_{m_1,\ldots,m_l=1}^{\Lambda} c_{m_1\sigma}^\dagger \ldots c_{m_l\sigma}^\dagger |0>, \quad l = 2, \ldots, \Lambda - 2 \tag{14}$$

when

$$V = \frac{\mu}{Z} - t - \frac{U}{2Z}, \quad J_z = 4(\frac{U}{2Z} - t - \frac{\mu}{Z}) \tag{15}$$

and, for $t \neq X$, $t \geq 0$, $Y - 2t \leq \frac{U}{Z} < -Y - 2t$, $\mu \geq 0$, $\frac{\mu}{Z} + t \geq X \geq -\frac{\mu}{Z}$, $\frac{J_{xy}}{4} \geq$ $max(-t - \frac{\mu}{Z} + \frac{U}{2Z}, \; -\frac{U}{2Z} + t + \frac{\mu}{Z} - 2(X-t)^2/(\frac{U}{Z} + Y + 2t))$ or $Y - 2t \leq \frac{U}{Z}$, $-Y - 2t < \frac{U}{Z}$, $\mu \geq 0$, $\frac{\mu}{Z} + t \geq X \geq -\frac{\mu}{Z}$, $-\frac{U}{2Z} + t + \frac{\mu}{Z} - 2(X-t)^2/(\frac{U}{Z} + Y + 2t) \geq \frac{J_{xy}}{4} \geq -t - \frac{\mu}{Z} + \frac{U}{2Z}$, or for $t = X \geq 0$,

$$-2t + |Y| \leq \frac{U}{Z} \leq 2t - \frac{|J_{xy}|}{2} + \frac{2\mu}{Z}, \quad \mu \geq 0. \tag{16}$$

The corresponding eigenvalue is $(\frac{\mu}{Z} - t)P$.

From formulae (1-4) one can construct more exact ground states such as $|\psi_6^\sigma> =$ $N\sum_{i=1}^{\Lambda}(-1)^{i+1}c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger c_{2,\downarrow}^\dagger \ldots c_{i\sigma}^\dagger \ldots c_{\Lambda,\uparrow}^\dagger c_{\Lambda,\downarrow}^\dagger |0>$, $|\psi_7^\sigma> = N\sum_{i=1}^{\Lambda}(-1)^{i+1}c_{i,\sigma}^\dagger|0>$, $|\psi_8^\sigma> =$ $N\sum_{i=1}^{\Lambda} c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger c_{2,\downarrow}^\dagger \ldots c_{i\sigma}^\dagger \ldots c_{\Lambda,\uparrow}^\dagger c_{\Lambda,\downarrow}^\dagger |0>$, $|\psi_9^\sigma> = N\sum_{i=1}^{\Lambda} c_{i,\sigma}^\dagger|0>$ in certain parameter regions. Some of the states have the same eigenvalues. For instance, the state $|\psi_6^\sigma>$ has the same eigenvalue as $|\psi_2^\sigma>$, however $|\psi_2^\sigma>$ and $|\psi_6^\sigma>$ correspond to different parameter regions. Thus in general the eigenvalues are not degenerate.

The h_{ij} in (6) have two different sets of eigenvectors and eigenvalues for $t = X$ and $t \neq X$ respectively. Therefore the parameter regions of the ground states have a jump when $t \to X$. In addition, all the ground states we presented above have no ODLRO. A ground state with ODLRO can be obtained from formula (5) by taking $|\xi_\alpha> = |0>$ and $|\xi_\beta> = c_\uparrow^\dagger c_\downarrow^\dagger |0>$. One then gets ground states that are equivalent to the η-pairing states with $\eta = 0$: $|\psi_l^{\eta=0}> = \sum_{m_1,\ldots,m_l=1}^{\Lambda} c_{m_1\uparrow}^\dagger c_{m_1\downarrow}^\dagger \ldots c_{m_l\uparrow}^\dagger c_{m_l\downarrow}^\dagger |0>$ within the parameter region $\frac{U}{Z} \leq min(-2|t| - 2V, \; \frac{J_z}{4} - V, \; -V - \frac{|J_{xy}|}{2} - \frac{J_z}{4})$, $t = X$, $Y = 2V$, $\mu = 0$ and $V \leq 0$.

Some properties of the phase diagram of the ground states can be analysed according to their related parameter regions. For instance, taking into account the conditions (15) and (16) (for simplicity we set $J_{xy} = 0$), we have that $|\psi_l^{\eta=0}>$ is a ground state for $\frac{U}{Z} \leq |2t|$. When $\frac{U}{Z} < -2t$, the η-pairing $|\psi_l^{\eta=0}>$ is no longer a ground state, but $|\phi_{2l}^\sigma>$ given by (14) is a ground state.

Some relations between the ground states above and the ground states studied in [4] can be discussed according to their parameter regions. As an example we set $t = X$, $Y = \mu = 0$, $V \geq 0$, and $J_z = -J_{xy} < 0$. It can be shown that for $|t| \geq J_{xy}/8$ the states $|\psi_1^\sigma >$ and $|F >$ are ground states in regions I and IV. While the state $|CDW >$ is a ground state in region II and III, see Fig. 1. More properties can be obtained when one changes the relations among the parameters. For instance, if $t = V$, then the states $|\psi_1^\sigma >$ and $|F >$ (resp. $|CDW >$) are ground states only in region I (resp. region II).
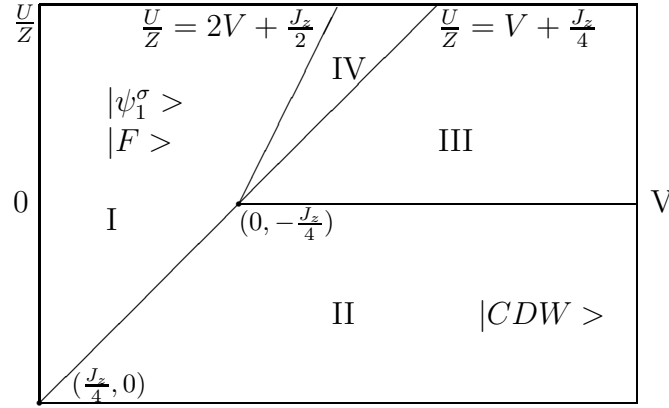

Figure 1: A ground state phase diagram related to states $|F >$, $|CDW >$ and $|\psi_1^\sigma >$

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